Algebra Through Problem Solving

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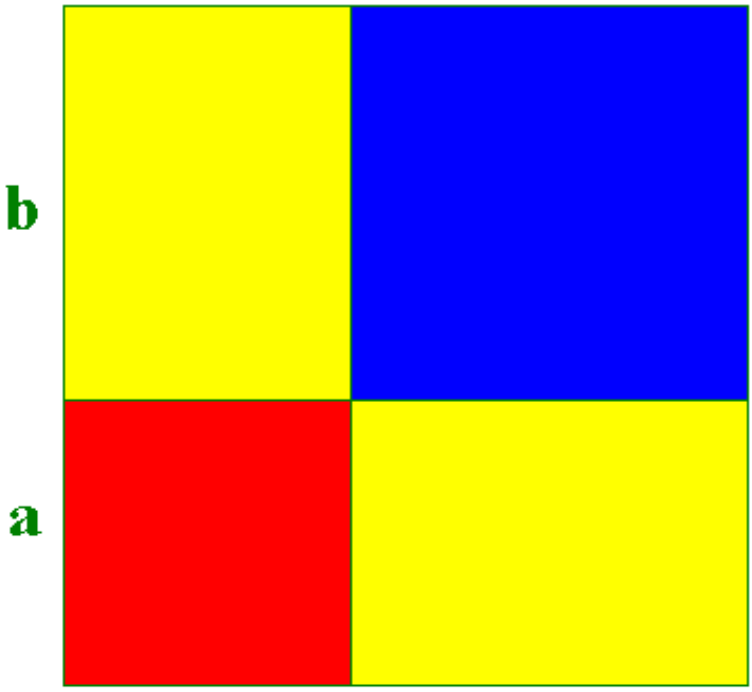

$$(a + b)^2 = ???$$

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To

Josephine Hillman

PREFACE

This book is an outgrowth of the authors' work in conducting problem solving seminars for undergraduates and high school teachers, in directing mathematics contests for undergraduates and high school students, and in the supervision of an undergraduate research participation program. Their experience has shown that interest in and knowledge of mathematics can be greatly strengthened by an opportunity to acquire some basic problem solving techniques and to apply these techniques to challenging problems for which the prerequisite knowledge is available.

Many students who have not had this opportunity lose confidence in themselves when they try unsuccessfully to solve non-routine problems such as those in the Mathematics Magazine or in the Putnam Intercollegiate Mathematics Competitions, conducted by the Mathematical Association of America. Those who gain self-confidence by work on challenging material at the proper level also generally have increased motivation for mastering significant mathematical concepts and for making original contributions to mathematical knowledge.

The topics chosen for this book are particularly appropriate since they are at a fairly elementary level and exhibit the interdependence of mathematical concepts. Many generalizations are suggested in the problems; the perceptive reader will be able to discover more.

The authors express their debt to all who have influenced this effort. We are especially grateful to Leonard Klosinski, Roseanna Torretto, and Josephine Hillman for their invaluable assistance.

Albuquerque, New Mexico — Abraham P. Hillman
Santa Clara, California — Gerald L. Alexanderson

TABLE OF CONTENTS

COVER PAGE i

COPYRIGHT ii

DEDICATION iii

PREFACE iv

INTRODUCTION vii

1. THE PASCAL TRIANGLE 1
Problems for Chapter1 5

2. THE FIBONACCI AND LUCAS NUMBERS 12
Problems for Chapter 2 14

3. FACTORIALS 18
Problems for Chapter 3 18

4. ARITHMETIC AND GEOMETRIC PROGRESSIONS 21
Problems for Chapter 4 23

5. MATHEMATICAL INDUCTION 29
Problems for Chapter 5 35

6. THE BINOMIAL THEOREM 40
Problems for Chapter 6 47

7. COMBINATIONS AND PERMUTATIONS 51
Problems for Chapter 7 55

8. POLYNOMIAL EQUATIONS 58
8.1 The Factor and Remainder Theorems 59
8.2 Integral Roots 63
8.3 Rational Roots 63
Problems for Sections 8.1, 8.2, and 8.3 64
8.4 Symmetric Functions 68
Problems for Section 8.4 68

9. DETERMINANTS 71
9.1 Determinants of Order 2 72
9.2 Determinants of Order 3 75
Problems for Sections 9.1 and 9.2 78
9.3 Determinants of Order n 81
Problems for Section 9.3 85
9.4 Vandermonde and Related Determinants 89
Problems for Section 9.4 90

10. INEQUALITIES 94
10.1 Elementary Properties 94
Problems for Section 10.1 95
10.2 Further Inequalities 97
Problems for Section 10.2 98
10.3 Inequalities and Means 99
10.4 The Cauchy-Schwarz Inequality 102
Problems for Sections 10.3 and 10.4 103

ANSWERS TO THE ODD-NUMBERED PROBLEMS 109

INDEX 117

INTRODUCTION

Knowledge of mathematics together with the ability to apply this knowledge to non-routine problems will be very valuable in the more and more automated world we face. Routine problems will tend to be solved mechanically, while new and challenging problems arise at a rapid rate for human minds to solve.

We hope that this book will be helpful in the process of mastering some aspects of mathematics and becoming proficient in using this knowledge. The topics selected contain important ideas that are often lost in the regular curriculum. An effort has been made to develop most of the theoretical material through sequences of related and progressively more sophisticated problems that follow the necessary text material and illustrative examples. In later chapters some proofs of a considerably more involved nature are omitted or dealt with only in special cases. The sets of problems provide opportunities for recognizing mathematical patterns and for conjecturing generalizations of specific results.

And now some words of advice to prospective problem solvers: some of the problems of this book are easy, some may take longer to solve than any previously encountered, and some may prove to be too difficult. If a problem is difficult, it may be helpful to look at the surrounding problems. One does not have to do all of the problems in a chapter before going on. In fact, some of the hard problems will appear to be easier if one returns to them after progressing through later chapters. The statements in problems preceded by the symbol "R" are required for subsequent work and should be specially noted.

Answers or hints are provided for the odd-numbered problems except starred problems and those that contain the answer in the problem. However, the greatest benefit comes from trying the problems; one should postpone looking at the answers as long as possible.

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